LECTURE 2: BESSEL'S AND LEGENDRE'S EQUATION

2.0 BESSEL'S EQUATION

This is a second order differential equation that occurs in engineering problems and it takes the form:

 $x^{2}y'' + xy' + (x^{2} - v^{2})y = 0$ where v is a constant which is a real number.

We proceed by choosing a series, $y = x^{c} [b_{0} + b_{1}x + b_{2}x^{2} + ... + b_{r}x^{r} + b_{r+1}x^{r+1} + ...]$

 $c = \pm v$ and $b_1 = 0$

The recurrence relation is

$$b_r = \frac{b_{r-2}}{v^2 - (c+r)^2}$$
 for $r \ge 2$

This shows that $b_1 = b_3 = b_5 = ... = 0$ noting that $b_1 = 0$

$$b_{2} = \frac{b_{0}}{v^{2} - (c+2)^{2}}; \quad b_{4} = \frac{b_{0}}{[v^{2} - (c+2)^{2}][v^{2} - (c+4)^{2}]}; \quad b_{6} = \frac{-b_{0}}{2^{6} \times 3!(v+1)(v+2)(v+3)}$$
$$b_{r} = \frac{(-1)^{\frac{r}{2}}b_{0}}{2^{r} \times (\frac{r}{2})!(v+1)(v+2)..(v+\frac{r}{2})} \text{ this is for all even } r.$$

If c= v, the resulting series solution becomes:

$$y = u = Ax^{\nu} \left[1 - \frac{x^2}{2^2(\nu+1)} + \frac{x^4}{2^4 \times 2!(\nu+1)(\nu+2)} - \frac{x^6}{2^6 \times 3!(\nu+1)(\nu+2)(\nu+3)} + \dots \right] + \frac{(-1)^{\frac{r}{2}}x^r}{2^r \times \left(\frac{r}{2}\right)!(\nu+1)(\nu+2)...(\nu+\frac{r}{2})} - \frac{x^6}{2^6 \times 3!(\nu+1)(\nu+2)(\nu+3)} + \dots \right]$$

This is valid provided v is not a negative integer.

When c = -v,

$$y = w = Bx^{-v} \begin{bmatrix} 1 + \frac{x^2}{2^2(v-1)} + \frac{x^4}{2^4 \times 2!(v-1)(v-2)} + \frac{x^6}{2^6 \times 3!(v-1)(v-2)(v-3)} + \dots \\ + \frac{(1)^{\frac{r}{2}}x^r}{2^r \times (\frac{r}{2})!(v-1)(v-2)\dots(v-\frac{r}{2})} \end{bmatrix}$$

This expression above is valid provided v is not a positive integer.

The complete Bessel's equation solution becomes:

$$y = u + w.$$

$$y = u + w = Ax^{v} \begin{bmatrix} 1 - \frac{x^{2}}{2^{2}(v+1)} + \frac{x^{4}}{2^{4} \times 2!(v+1)(v+2)} - \frac{x^{6}}{2^{6} \times 3!(v+1)(v+2)(v+3)} + \dots \\ + \frac{(-1)^{\frac{r}{2}}x^{r}}{2^{r} \times (\frac{r}{2})!(v+1)(v+2)..(v+\frac{r}{2})} \end{bmatrix} + Bx^{-v} \begin{bmatrix} 1 + \frac{x^{2}}{2^{2}(v-1)} + \frac{x^{4}}{2^{4} \times 2!(v-1)(v-2)} + \frac{x^{6}}{2^{6} \times 3!(v-1)(v-2)(v-3)} + \dots \\ + \frac{(1)^{\frac{r}{2}}x^{r}}{2^{r} \times (\frac{r}{2})!(v-1)(v-2)..(v-\frac{r}{2})} \end{bmatrix}$$

2.1 BESSEL FUNCTION

The obtained result above can be presented in terms of gamma functions. This applies to x>0.

$$\Gamma(x+1) = x\Gamma(x)$$

$$\Gamma(x+2) = (x+1)\Gamma(x+1) = (x+1)x\Gamma(x)$$

$$\Gamma(x+3) = (x+2)\Gamma(x+2) = (x+2)(x+1)x\Gamma(x)$$

If the arbitrary constant is assigned b_0 value, $\frac{1}{2^{\nu}\Gamma(\nu+1)}$, then for c = v,

Recall,

$$b_{2} = \frac{b_{0}}{v^{2} - (c+2)^{2}} = \frac{b_{0}}{(v-c-2)(v+c+2)} = \frac{b_{0}}{-2(2v+2)}$$
$$= \frac{1}{-2(2v+2)} \bullet \frac{1}{2^{v} \Gamma(v+1)} = \frac{-1}{2^{v+2}(v+1)\Gamma(v+1)} = \frac{-1}{2^{v+2}(1!)\Gamma(v+2)}$$

$$b_4 = \frac{b_2}{[v^2 - (c+4)^2]} = \frac{b_2}{[v - c - 4][v + c + 4]} = \frac{b_2}{-2^3(v+2)}$$
$$= \frac{b_2}{-2^3(v+2)} = \frac{-1}{2^3(v+2)} \bullet \frac{-1}{2^{v+2}(1!)\Gamma(v+2)} = \frac{1}{2^{v+4}(2!)\Gamma(v+3)}$$

It can be observed that the coefficients of the series take a pattern of the expression:

$$b_r = \frac{(-1)^{\frac{r}{2}}}{2^{\nu+r}(\frac{r}{2}!)\Gamma\left(\nu+\frac{r}{2}+1\right)} \text{ for all even r.}$$
$$\Gamma\left(\nu+\frac{r}{2}+1\right) = \left(\nu+\frac{r}{2}\right)!$$

For r = 2k

$$b_{2k} = \frac{(-1)^k}{2^{\nu+2k}(k!)\Gamma(\nu+k+1)}$$
 where k= 1, 2,3,....

The series takes the form:

$$y = x^{\nu} \left[\frac{1}{2^{\nu} \Gamma(\nu+1)} - \frac{x^2}{2^{\nu+2} (1!) \Gamma(\nu+2)} + \frac{x^4}{2^{\nu+4} (2!) \Gamma(\nu+3)} + \dots + \frac{(-1)^k x^{2k}}{2^{\nu+2k} (k!) \Gamma(\nu+k+1)} \right]$$

This is called the Bessel function of the first kind of order v and it is denoted by $J_{v}(x)$ This can be rewritten as:

$$J_{\nu}(x) = \left(\frac{x}{2}\right)^{\nu} \left[\frac{1}{\Gamma(\nu+1)} - \frac{x^2}{2^2(1!)\Gamma(\nu+2)} + \frac{x^4}{2^4(2!)\Gamma(\nu+3)} + \dots + \frac{(-1)^k x^{2k}}{2^{2k}(k!)\Gamma(\nu+k+1)}\right]$$

The expression above applies provided that v is a positive integer.

When v is a negative integer (c = -v), then the governing expression becomes:

$$J_{-\nu}(x) = \left(\frac{x}{2}\right)^{-\nu} \left[\frac{1}{\Gamma(1-\nu)} - \frac{x^2}{2(1!)\Gamma(2-\nu)} + \frac{x^4}{2^2(2!)\Gamma(3-\nu)} + \dots + \frac{(-1)^k x^{2k}}{2^{2k} (k!)\Gamma(k-\nu+1)}\right]$$

The complete Bessel function can be expressed generally as:

$$J_{\nu}(x) = \left(\frac{x}{2}\right)^{\nu} \sum_{k=0}^{\infty} \frac{(-1)^{k} x^{2k}}{2^{2k} (k!) \Gamma(\nu + k + 1)}$$
$$J_{-\nu}(x) = \left(\frac{x}{2}\right)^{-\nu} \sum_{k=0}^{\infty} \frac{(-1)^{k} x^{2k}}{2^{2k} (k!) \Gamma(k - \nu + 1)}$$

The convergence of the series for all values of x can be established by normal ratio test.

 $J_{\nu}(x)$ and $J_{-\nu}(x)$ are two independent solutions of the original equation, therefore, the complete solution becomes:

 $y = AJ_{v}(x) + BJ_{-v}(x)$

Where A and B are constants.

Some Bessel functions are commonly used and are worthy of special note. When v is a positive integer which is denoted by n, we have:

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k} (k!) \Gamma(n+k+1)}$$

Note that $\Gamma(n+k+1) = (n+k)!$ for k= 0, 1, 2, 3,....

The above equation can be rewritten as:

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k} (k!)(n+k)!}$$

When v =n, $J_n(x)$ and $J_{-n}(x)$ are not independent but have a relationship in the form:

$$J_{-n}(x) = (-1)^n J_n(x)$$

Recall that $\Gamma(x+1) = x\Gamma(x)$

Therefore,

$$\Gamma(x) = \frac{\Gamma(x+1)}{x}$$

For the value of zero for x or a negative integral value, $\Gamma(x)$ is infinite.

$$J_{-\nu}(x) = \left(\frac{x}{2}\right)^{-\nu} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k} (k!) \Gamma(k-\nu+1)}$$

When consider the gamma function $\Gamma(k-v+1)$ in the denominator, as v tends closely to a positive integer n, as shown: $\Gamma(k-v+1) \rightarrow \Gamma(k-n+1)$. The first finite value of $\Gamma(k-n+1)$ occurs at k = n.

When the values of $\Gamma(k-v+1)$ are infinite, the coefficients of $J_{-v}(x)$ are zero. Therefore, the series starts at k= n.

$$J_{-n}(x) = \left(\frac{x}{2}\right)^{-n} \sum_{k=n}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k} (k!) \Gamma(k-n+1)} = \sum_{k=n}^{\infty} \frac{(-1)^k x^{2k-n}}{2^{2k-n} (k!) \Gamma(k-n+1)}$$

Let's take k = p+n.

We then have,

$$J_{-n}(x) = \sum_{p=0}^{\infty} \frac{(-1)^{p+n} x^{2p+n}}{2^{2p+n} (k!)(k-n)!}$$

Note: k-n = p

$$= (-1)^{n} \sum_{p=0}^{\infty} \frac{(-1)^{p} x^{2p+n}}{2^{2p+n} (p+n)! (p!)}$$
$$= (-1)^{n} \left(\frac{x}{2}\right)^{n} \sum_{p=0}^{\infty} \frac{(-1)^{p} x^{2p+n}}{2^{2p} (p!) (p+n)!}$$
$$= (-1)^{n} \left(\frac{x}{2}\right)^{n} \sum_{k=0}^{\infty} \frac{(-1)^{k} x^{2k}}{2^{2k} (k!) (k+n)!}$$

Therefore,

$$J_{-n}(x) = (-1)^n J_n(x)$$

The series of for $J_n(x)$

$$J_n(x) = \left(\frac{x}{2}\right)^n \left[\frac{1}{n!} - \frac{1}{(n+1)!} \left(\frac{x}{2}\right)^2 + \frac{1}{(2!)(n+2)!} \left(\frac{x}{2}\right)^4 - \dots\right]$$

We can obtain two commonly used functions from the above expression.

$$J_0(x) = 1 - \frac{1}{(1!)^2} \left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2} \left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2} \left(\frac{x}{2}\right)^6 + \dots$$

And

$$J_1(x) = \frac{x}{2} \left[1 - \frac{1}{(1!)(2!)} \left(\frac{x}{2}\right)^2 + \frac{1}{(2!)(3!)} \left(\frac{x}{2}\right)^4 + \dots \right]$$



Figure 2.0: Graphs of Bessel functions $J_0(x)$ and $J_1(x)$

2.2 LEGENDRE'S EQUATION

Legendre's equation forms of equations are of special interest in engineering application. $(1-x^2)y'' - 2xy' + k(k+1)y = 0$ where k is a real constant.

The above equation can be solved by Frobenius method. The indicial equation gives c=0 and c=1, and the two corresponding solutions are:

(a)
$$c = 0; y = a_0 \left[1 - \frac{k(k+1)}{2!} x^2 + \frac{k(k-2)(k+1)(k+3)}{4!} x^4 - ... \right]$$
 even powers of x
(b) $c = 1; y = a_1 \left[x - \frac{(k-1)(k+2)}{3!} x^3 + \frac{(k-1)(k-3)(k+2)(k+4)}{5!} x^5 - ... \right]$ odd powers of x

where a_0 and a_1 are arbitrary constants.

2.3 LEGENDRE POLYNOMIALS

In a situation where k is an integer (n), the solution series terminates after a finite number of terms. The resulting polynomial in x, denoted by $P_n(x)$, is called a Legendre polynomial, with a_0 or a_1 being chosen so that the polynomial has a unit when x =1.

For example, to obtain $P_2(x)$ from the equation, we know that n = k = 2.

Recall,

$$y = a_0 \left[1 - \frac{k(k+1)}{2!} x^2 + \frac{k(k-2)(k+1)(k+3)}{4!} x^4 - \dots \right]$$

Therefore,

$$y = a_0 \left[1 - \frac{2(2+1)}{2!} x^2 + \frac{2(2-2)(2+1)(2+3)}{4!} x^4 - \dots \right]$$
$$y = a_0 \left[1 - \frac{2 \times 3}{2!} x^2 + 0 \dots \right] = a_0 \left(1 - 3x^2 \right)$$

The constant a_0 is chosen that makes y = 1 when x = 1.

That is:
$$1 = a_0(1-3)$$

This gives $a_0 = -\frac{1}{2}$

Therefore,

$$P_2(x) = -\frac{1}{2}(1 - 3x^2) = \frac{1}{2}(3x^2 - 1)$$

When n = k = 3

$$y = a_1 \left[x - \frac{(k-1)(k+2)}{3!} x^3 + \frac{(k-1)(k-3)(k+2)(k+4)}{5!} x^5 - \dots \right]$$

We now have:

$$y = a_1 \left[x - \frac{(3-1)(3+2)}{3!} x^3 + \frac{(3-1)(3-3)(3+2)(3+4)}{5!} x^5 - \dots \right]$$

This gives:

$$y = a_1 \left[x - \frac{(2)(5)}{3!} x^3 + 0 \right] = a_1 \left(x - \frac{5}{3} x^3 \right)$$

Taking y = 1 when x = 1, the above equation becomes,

$$1 = a_1 \left(1 - \frac{5}{3} \right)$$

Therefore,

$$a_1 = -\frac{3}{2}$$

$$P_3(x) = -\frac{3}{2}\left(x - \frac{5}{3}x^3\right) = \frac{1}{2}\left(5x^3 - 3x\right)$$

QUESTION 1

Express $f(x) = x^3 + 3x^2 + x - 5$ in terms of Legendre's polynomials.

Solution

Using Rodrigue's formula,

$$P_{n}(x) = \frac{1}{2^{n} n!} \frac{d^{n}}{dx^{n}} (x^{2} - 1)^{n}$$
(i) for n = 0,1,2,3,4, ...

$$P_{0}(x) = 1$$
(ii)

$$P_{1}(x) = x$$
(iii)

$$P_{2}(x) = \frac{(3x^{2} - 1)}{2}$$
(iv)

$$P_{3}(x) = \frac{(5x^{3} - 3x)}{2}$$
(v)

From equation (iv), x^2 is made the subject of formula. We get:

$$x^{2} = (2P_{2}(x)+1) \cdot \frac{1}{3} = \frac{2}{3}P_{2}(x) + \frac{1}{3}$$

Recall,

$$P_0(x) = 1$$

Therefore,

$$x^{2} = \frac{2}{3}P_{2}(x) + \frac{P_{0}(x)}{3}$$

From equation (v), x^3 is made the subject of formula. We get:

$$x^{3} = \frac{2}{5}P_{3}(x) + \frac{3P_{1}(x)}{5}$$

Substitute the values of x in equation (i).

$$f(x) = x^{3} + 3x^{2} + x - 5$$
(i)

$$f(x) = \left(\frac{2}{5}P_{3}(x) + \frac{3P_{1}(x)}{5}\right) + 3\left(\frac{2}{3}P_{2}(x) + \frac{P_{0}(x)}{3}\right) + P_{1}(x) - 5P_{0}(x)$$

$$f(x) = \frac{2}{5}P_{3}(x) + 2P_{2}(x) + \frac{8P_{1}(x)}{5} - 4P_{0}(x)$$

2.4 RODRIGUE'S FORMULA AND THE GENERATING FUNCTION

Legendre polynomials can be derived by Rodrigue's formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

Solution

Let
$$v = (x^2 - 1)^n$$
 and $u = (x^2 - 1)$

Therefore,

 $v = u^n$

The nth derivative of u is a solution of the Legendre's differential equation

$$(1-x^{2})y'' - 2xy' + n(n+1)y = 0$$

Differentiating with respect to x, using product rule, we have:

$$v_{x} = \frac{dv}{dx} = \frac{du}{dx} \cdot \frac{dv}{du} = 2x \cdot n(x^{2} - 1)^{n-1} = \frac{2xn(x^{2} - 1)^{n}}{(x^{2} - 1)}$$

This can be rewritten as:

$$(x^2-1)v_x = 2xn(x^2-1)^n$$

Differentiating the above with respect to x gives:

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$
$$(x^{2} - 1)v_{xx} + 2xv_{x} = 2n \cdot 2xn(x^{2} - 1)^{n-1} + 2n(x^{2} - 1)^{n} = 2n(2xn(x^{2} - 1)^{n-1} + (x^{2} - 1)^{n})$$

Recall,

$$v = (x^2 - 1)^n$$
 and $v_x = 2x \cdot n(x^2 - 1)^{n-1}$

This can be rewritten as:

$$(x^{2} - 1)v_{xx} + 2xv_{x} = 2n(xv_{x} + v)$$

To differentiate the above expression n times, Leibnitz theorem is applied. For the nth derivative, the expression is given as:

$$(RW)_n = RW_n + nR_xW_{n-1} + \frac{n(n-1)}{2!}R_{xx}W_{n-2} + \dots + R_nW$$

Applying the above expression, we can write the expression as:

$$\left[\left(x^{2} - 1 \right) v_{xx} \right]_{n} + 2 \left[x v_{x} \right]_{n} = 2n \left[x v_{x} \right]_{n} + 2nv_{n}$$

$$\left[\left(x^{2} - 1 \right) v_{n+2} + n \cdot 2x \cdot v_{n+1} + \frac{n(n-1) \cdot 2 \cdot v_{n}}{2} \right] + 2 \left[x v_{n+1} + n(1)v_{n} \right] = 2n \left[x v_{n+1} + n(1)v_{n} \right] + 2nv_{n}$$

$$\left(x^{2} - 1 \right) v_{n+2} + 2xnv_{n+1} + n(n-1)v_{n} + 2xv_{n+1} + 2nv_{n} = 2nxv_{n+1} + 2n^{2}v_{n} + 2nv_{n}$$

$$(x^{2} - 1)v_{n+2} + (2xn + 2x - 2nx)v_{n+1} + (n(n-1) + 2n - 2n^{2} - 2n)v_{n} = 0$$

$$(x^{2} - 1)v_{n+2} + (2x)v_{n+1} - n(n+1)v_{n} = 0$$

$$(1 - x^{2})v_{n+2} - (2x)v_{n+1} + n(n+1)v_{n} = 0$$